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The elastostatic approach to compression of a cracked medium with dry friction

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Summary

In the frame of a linear elastic material compression of a cracked plane is considered. Friction during a mutual sliding of the crack surfaces can be responsible for some non-linear effects, in particular, for hysteresis. A rigorous solution to the problem of non-axisymmetric compression of a space weakened by a circular crack with dry friction is given. This solution is obtained in displacements and the field of displacements is represented in elementary functions.

1. Introduction

Compression of an elastic plane weakened by a rectilinear slit with the presence of dry friction was studied in [1]. By now the contact problem for the elastic plane with a cut is well understood [2]. The solution of contact problems with allowance for friction depends essentially on the order of application of the external loads [3]. In Section 2 of the present paper the influence of external-loading history is investigated for the stress-strain state of an infinite plane with a crack. It is shown quantitatively that the presence of friction between the crack surfaces can be responsible for hysteresis of the cracked medium. An elastic space with a circular slit which is compressed by a uniformly distributed load applied in the axial direction was investigated in [4]. In Section 3 of the present paper non-axisymmetric compression of the space with a circular crack is studied. After this the ideas of Section 2 can be extended to the three-dimensional case.

2. Influence of external-loading history on stress-strain state of a cracked medium in the presence of friction

2.1. External-loading history consideration

We consider the following problem. An elastic plane is weakened by a mathematical cut along the real-axis segment $|x| \leqslant 1$. The plane is subjected at infinity to the action of a

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uniformly distributed shear load $X_{y1}^{\infty} = T$. Under this load the crack contour is free of external stresses. The complex potentials $\phi(z)$ and $\Omega(z)$ associated with the stresses and displacements by the Muskhelishvili formulae take the following form in this case [5]:

$$\phi_1(z) = -iT(z^2 - l^2)^{-1/2} z/2 + iT/2,$$

$$\Omega_1(z) = iT(z^2 - l^2)^{-1/2} z/2 - iT/2.$$

Now let an additional uniformly distributed compressive load $Y_{y2}^{\infty} = -P$ be applied. The solution to the problem of the complex loading at infinity, ϕ_{12} and Ω_{12}, may be represented in the form of a sum $\phi_{12} = \phi_1 + \phi_2$, $\Omega_{12} = \Omega_1 + \Omega_2$; we seek ϕ_2 and Ω_2 from the boundary conditions

$$(Y_{y2} - iX_{y2})^+ = (Y_{y2} - iX_{y2})^-, \qquad (u_2' + iv_2')^+ = (u_2' + iv_2')^-, \qquad |x| < 1, \tag{1}$$

where + and − denote the limiting values of a function. The crack surfaces are shifted owing to shear, but we neglect this in writing the boundary conditions (1); we prescribe the boundary conditions on the undeformed contour. It is evident that $\phi_2(z) = -P/4$, $\Omega_2(z) = -3P/4$.

On the crack contour

$$Y_{y12}^{\pm} = -P, \qquad X_{y12}^{\pm} = 0, \qquad u_{12}^{+} - u_{12}^{-} = \frac{T(\kappa+1)}{2\mu}(l^2 - x^2)^{1/2},$$

where $\kappa = 3 - 4\nu$, ν is Poisson's ratio.

Let us consider the problem with the reverse order of external-load application. Since the crack is perpendicular to the line of action of the compressive forces, the boundary conditions at the first stage will be adhesion conditions of the form (1). When the shear load T is applied, the adhesion condition for the crack surfaces is preserved if $T < fP$, where f is the friction coefficient. In this case

$$\phi_{34} = -P/4, \qquad \Omega_{34} = -3P/4 - iT, \qquad Y_{y34}^{\pm} = -P, \qquad X_{y34}^{\pm} = T, \qquad u_{34}^{+} - u_{34}^{-} = 0.$$

If $T > fP$, then the crack surfaces slide past one another and the solution may be represented in the form

$$\phi_{345} = \lim_{T \to fP} \phi_{34} + \phi_5, \qquad \Omega_{345} = \lim_{T \to fP} \Omega_{34} + \Omega_5,$$

where ϕ_5 and Ω_5 are the solution to the following problem:

$$(Y_{y5} - iX_{y5})^+ = (Y_{y5} - iX_{y5})^-, \qquad X_{y5}^{+} = -fY_{y5}^{+}, \qquad v_5'^{+} = v_5'^{-}, \qquad |x| < 1. \tag{2}$$

The boundary conditions (2) lead to the boundary-value problem for potentials $\phi_5(z)$ and $\Omega_5(z)$:

$$[\phi_5 - \bar{\phi}_5 - \Omega_5 + \bar{\Omega}_5]^+ = [\phi_5 - \bar{\phi}_5 - \Omega_5 + \bar{\Omega}_5]^-,$$

$$[\phi_5 + \bar{\phi}_5 - \Omega_5 - \bar{\Omega}_5]^+ = [\phi_5 + \bar{\phi}_5 - \Omega_5 - \bar{\Omega}_5]^-,$$

$$[\kappa\phi_5 + \kappa\bar{\phi}_5 + \Omega_5 + \bar{\Omega}_5]^+ = [\kappa\phi_5 + \kappa\bar{\phi}_5 + \Omega_5 + \bar{\Omega}_5]^-,$$

$$[(i+f)\phi_5 - (i-f)\bar{\Omega}_5]^+ = [(i-f)\bar{\phi}_5 - (i+f)\Omega_5]^-, |x| < 1. \tag{3}$$

The first three equations of this system are called by Muskhelishvili homogeneous problems of linear conjugation. Multiplying the first equation or (3) by $-f$, the second by $-i$, the fourth by two, and adding, we obtain a fourth linear conjugation problem

$$[(i+f)\phi_5 + (f-i)\bar{\phi}_5 + (f+i)\Omega_5 + (f-i)\bar{\Omega}_5]^+$$
$$= -[(i+f)\phi_5 + (f-i)\bar{\phi}_5 + (f+i)\Omega_5 + (f-i)\bar{\Omega}_5]^-. \tag{4}$$

Taking the relationships $\phi_5(\infty) = 0$, $\Omega_5(\infty) = -i(T - fP)$ into account, we find from (3) and (4):

$$\phi_{345} = -P/4 + i(T - fP)\left(1 - z(z^2 - l^2)^{-1/2}\right)/2,$$
$$\Omega_{345} = -3P/4 - ifP - i(T - fP)\left(1 + z(z^2 - l^2)^{-1/2}\right)/2. \qquad \text{(for } T > fP\text{)}$$

On the crack contour

$$Y_{y345}^{\pm} = -P, \qquad X_{y345}^{\pm} = fP, \qquad u_{345}^{+} - u_{345}^{-} = (T - fP)\frac{(\kappa + 1)}{2\mu}(l^2 - x^2)^{1/2}.$$

We note that in the given case a change in the order of external-load application led to a change in the stress and strain fields in the elastic plane. In particular, tangential stresses appeared on the crack surfaces, reducing the mutual displacement of the surfaces.

2.2. Compression-unloading of a plane with a crack

Let us consider the compression of an elastic plane containing a crack $2l$ long, forming an angle β ($0 < \beta < \pi/2$) with the direction of action of the external load q. Depending on the orientation of the crack, on its contour there will be realized either surface adhesion conditions of the form (1) or sliding conditions of the form (2) [2]. We construct the solution of this problem in analogy with the preceding treatment. In the first case on the crack contour

$$Y_{y1}^{\pm} = -\frac{q}{2}(1 - \cos 2\beta), \qquad X_{y1}^{\pm} = \frac{q}{2}\sin 2\beta, \qquad u_1^+ - u_1^- = 0,$$

while in the second

$$Y_{y1}^{\pm} = -\frac{q}{2}(1 - \cos 2\beta), \qquad X_{y1}^{\pm} = f\frac{q}{2}(1 - \cos 2\beta),$$

$$(u_1^+ - u_1^-) = \frac{\kappa + 1}{4\mu}q[\sin 2\beta - f(1 - \cos 2\beta)](l^2 - x^2)^{1/2}.$$

It is not difficult to show that sliding of the crack surfaces occurs in the $0 < \beta < \operatorname{arctg}(1/f)$

range and adhesion in the $\operatorname{arctg}(1/f) \leqslant \beta \leqslant \pi/2$ range. After the load has reached a certain value q_0, let unloading begin. We construct the solution by the method indicated in 2.1. At the beginning of the unloading process there will be adhesion of the crack surfaces regardless of the crack orientation since the shear stress should change sign, i.e.

$$Y_{y12}^{\pm} = -\frac{q}{2}(1-\cos 2\beta), \qquad X_{y12}^{\pm} = \frac{q}{2}\sin 2\beta,$$

$$u_{12}^{+} - u_{12}^{-} = 0, \qquad \text{for } \beta \in [\operatorname{arctg}(1/f), \pi/2]. \tag{5}$$

For the case in which $\beta \in [0, \operatorname{arctg}(1/f)]$,

$$Y_{y12}^{\pm} = -\frac{q}{2}(1-\cos 2\beta), \qquad X_{y12}^{\pm} = f\frac{q_0}{2}(1-\cos 2\beta) + \frac{(q-q_0)}{2}\sin 2\beta,$$

$$u_{12}^{+} - u_{12}^{-} = \frac{\kappa+1}{4\mu} q_0 [\sin 2\beta - f(1-\cos 2\beta)](l^2-x^2)^{1/2}. \tag{6}$$

With a further increase in load, slippage of opposite sign will appear. Using (6), from the condition $X_{y12}^{\pm} = fY_{y12}^{\pm}$ we obtain the limits of the interval of renewed slippage

$$0 < \beta < \operatorname{arctg}(1/A), \qquad A = f(q_0+q)(q_0-q)^{-1}. \tag{7}$$

In this interval the following conditions are satisfied on the crack contour:

$$(Y_{y3} - iX_{y3})^{+} = (Y_{y3} - iX_{y3})^{-}, \qquad X_{y3}^{+} = fY_{y3}^{+}, \qquad v_3'^{+} = v_3'^{-}, \qquad |x| < 1.$$

Solving the problem, we obtain on the crack contour

$$Y_{y3}^{\pm} = -\tfrac{1}{2}(q-q_*)(1-\cos 2\beta), \qquad X_{y3}^{\pm} = -\tfrac{1}{2}f(q-q_*)(1-\cos 2\beta),$$

$$u_3^{+} - u_3^{-} = \frac{\kappa+1}{4\mu}(q-q_*)[\sin 2\beta + f(1-\cos 2\beta)](l^2-x^2)^{1/2}, \tag{8}$$

$$q_* = \frac{q_0(1-f\operatorname{tg}\beta)}{(1+f\operatorname{tg}\beta)}.$$

Taking (6) into account, in the renewed slippage range we have

$$Y_{y123}^{\pm} = -\tfrac{1}{2}q(1-\cos 2\beta), \qquad X_{y123}^{\pm} = -\tfrac{1}{2}fq(1-\cos 2\beta),$$

$$u_{123}^{+} - u_{123}^{-} = \frac{\kappa+1}{4\mu} q[\sin 2\beta + f(1-\cos 2\beta)](l^2-x^2)^{1/2}.$$

For $\operatorname{arctg}(1/A) \leqslant \beta \leqslant \operatorname{arctg}(1/f)$, the stresses and displacements on the crack contour are calculated from (6), and for $\operatorname{arctg}(1/f) \leqslant \beta \leqslant \pi/2$ from (5). After load removal, the elastic plane returns to the initial state regardless of the crack orientation.

2.3. On hysteresis of the cracked medium

Let us use the results obtained in 2.2 for explanation of the hysteresis of the cracked medium.

Let an elastic plane contain cracks having a certain characteristic length $2l$, having equiprobable distribution over direction. We assume that crack concentration is small and that we may neglect the mutual influence of the cracks. The plane is subjected to uniaxial compression by a uniformly distributed load applied at infinity. We note that the method proposed here makes it possible to solve analogous problems for other cracked-body loading cases as well. We isolate a typical square having side $L \gg 1$, containing a single crack of arbitrary orientation. For it we determine the effective Young's modulus for loading and for unloading. This question has been investigated for loading in [6].

We consider two stress-strain states for the isolated square (Figure 1): the first is actual loading with slippage of the crack surfaces and the second is loading by forces that ensure adhesion of the surfaces. Applying the theorem on the reciprocity of work, we find:

$$qL\Delta_p = pL\Delta_q - \int_{-l}^{l} \tau(u_1^+ - u_1^-)\mathrm{d}x. \tag{9}$$

The effective Young's modulus $E_{\beta 1}$ is realized for the first state, and the intrinsic modulus of elasticity E for the second. As a consequence, $\Delta_p = pL/E$, $\Delta_q = qL/E_{\beta 1}$, and from (9) we have

$$\frac{1}{E_{\beta 1}} = \frac{1}{E} + \int_{-l}^{l} \frac{\tau(u_1^+ - u_1^-)}{pqL^2}\mathrm{d}x, \qquad \tau = p \sin\beta \cos\beta. \tag{10}$$

We finally obtain

$$E_{\beta 1} = E\left[1 + \frac{2\pi l^2}{L^2} \sin\beta \cos\beta \left(\sin\beta \cos\beta - f \sin^2\beta\right)\right]^{-1}. \tag{11}$$

For unloading from q_0 to q_* adhesion of the crack surfaces is realized. At this stage the effective elastic modulus of the square coincides with the intrinsic modulus of the material. Then sliding of the surfaces is renewed. This process is described by (8), but now we take as the first state the action on the square (unloaded to force q_*) of a load of opposite sign, causing renewed slippage of the surfaces. Here the effective Young's modulus $E_{\beta u}$ is realized. As the second state we consider the same loading as in the first case. Application

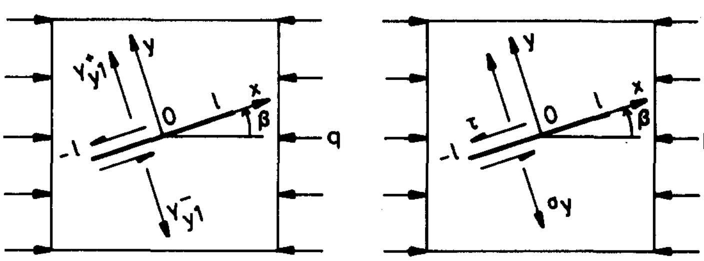

Fig. 1. Application of the theorem on the reciprocity of work.

of the work reciprocity theorem in this case leads to the relationship

$$\frac{1}{E_{\beta u}} = \frac{1}{E} + \int_{-l}^{l} \frac{\tau(u_3^+ - u_3^-)}{p(q - q_*)L^2} \mathrm{d}x. \tag{12}$$

Taking into account (8) and (12), we find

$$E_{\beta u} = E\left[1 + \frac{2\pi l^2}{L^2} \sin\beta \cos\beta \left(\sin\beta\cos\beta + f\sin^2\beta\right)\right]^{-1}. \tag{13}$$

We further assume that a large number N of the squares just considered are located one after the other to form a long strip; there is an equiprobable distribution of crack directions in the squares. The average strain for this strip under longitudinal loading by a force q will be ($a = \mathrm{arctg}(1/f)$)

$$\langle\epsilon_l(q)\rangle = \frac{q}{N}\left[\int_0^a \frac{1}{E}\left[1 + \frac{\pi l^2}{L^2} \sin\beta\cos\beta\left(\sin\beta\cos\beta - f\sin^2\beta\right)\right]\mathrm{d}\beta\frac{2N}{\pi} + \int_a^{\pi/2} \frac{1}{E}\mathrm{d}\beta\frac{2N}{\pi}\right].$$

Here summation is replaced by integration since the average step for crack-direction variation is small. Evaluating the integrals [7], we obtain

$$\langle\epsilon_l\rangle = \frac{q}{E}\left[1 + \frac{l^2}{2L^2}\left(\mathrm{arctg}\,\frac{1}{f} - \frac{f}{1+f^2}\right)\right].$$

As a consequence, the effective elastic modulus for the cracked medium under loading is calculated from the formula

$$\langle E_l\rangle = E\left[1 + \frac{l^2}{2L^2}\left(\mathrm{arctg}\,\frac{1}{f} - \frac{f}{1+f^2}\right)\right]^{-1}.$$

When $f \to \infty$, we have $\langle E_l\rangle = E$, which agrees with the physical considerations. Arguing in similar fashion and using (7), (13), for the case of unloading of the plane we obtain

$$\langle E_u\rangle = E\left[1 + \frac{l^2}{2L^2}\left(\mathrm{arctg}\,\frac{1}{A} - \frac{A}{1+A^2} + \frac{4fq_0}{(q_0 - q)(1+A^2)^2}\right)\right]^{-1}.$$

For $q = q_0$, we have $\langle E_u\rangle = E$, i.e. the intrinsic Young's modulus of the material may be found from the slope of the $\langle\epsilon_u\rangle = \langle\epsilon_u(q)\rangle$ curve at the initial instant of unloading. This known experimental fact was predicted qualitatively in [6].

The strain for the cracked medium under unloading is determined from the formula

$$\langle \epsilon_u \rangle = \langle \epsilon_l \rangle - \int_q^{q_0} \frac{\mathrm{d}q}{\langle E_u(q) \rangle} = \frac{q}{E} + \frac{l^2}{2EL^2}\left[q_0 \left(\operatorname{arctg}\frac{1}{f} - \frac{f}{1+f^2} \right) \right.$$

$$\left. - \frac{2fg_0}{1+f^2}\left[-\frac{1+fA}{1+A^2} + \frac{q_0-q}{2q_0} + \frac{(q_0-q)(1+f^2)}{2fq_0} \operatorname{arctg}\frac{1}{A} \right] \right].$$

The $\langle \epsilon \rangle \sim q$ diagram for a compression-unloading cycle takes the form of a loop running clockwise. When the load rises from 0 to some value q_0, the diagram is a straight line $q = \langle E \rangle \langle \epsilon \rangle$, $\langle E \rangle < E$. With unloading, the effective elastic modulus depends on the magnitude of the external load. The intrinsic modulus of the material is realized at the initial instant of unloading; when the load is removed the cracked plane returns to the initial undeformed state. The area bounded by the $q = q(\langle \epsilon \rangle)$ curve represents the specific energy dissipation of the cracked plane over a single compression-unloading cycle.

3. Nonaxisymmetric compression of elastic space with a circular crack

3.1. Statement of the 3D-problem

We consider a linear elastic isotropic space with circular mathematical crack S, given by conditions

$$x_3 = 0, \qquad r < a, \tag{14}$$

where a is radius of the crack, $r^2 = x_1^2 + x_2^2$. Let the space be compressed uniaxially by a constant load $-p$, applied at infinity to an area, defined by a normal (quite generally) $\mathbf{n} = \{0, \sin\beta, \cos\beta\}$, where β is an angle between the compression direction and x_3-axis. The value of this angle is supposed to be such one that a mutual sliding of the crack surfaces takes place. Tangential and normal contact stresses are bounded by Coulomb's law. Coefficient of friction is denoted by f.

We construct the solution of this problem by the method of superposition. Firstly, as in [8], a simple problem of elastostatics about the compression of continuous space by the former load should be solved. When elastic material is clamped at the origin, displacements to the directions of x_1, x_2, x_3 axes are

$$u_1^0 = \frac{p\nu x_1}{2\mu(1+\nu)},$$

$$u_2^0 = \frac{-p}{2\mu(1+\nu)}\{[1-(1+\nu)\cos^2\beta]x_2 + (1+\nu)\sin\beta\cos\beta x_3\},$$

$$u_3^0 = \frac{p}{2\mu(1+\nu)}\{-(1+\nu)\sin\beta\cos\beta x_2 + [-1+(1+\nu)\sin^2\beta]x_3\}, \tag{15}$$

where μ is shear modulus, ν is Poisson's ratio. Then using general Hooke's law and

Cauchy's geometrical relations, we find the stress vector in the place of the crack S (14):

$$\mathbf{t}^0(x_1, x_2, 0) = \begin{bmatrix} \sigma_{13}^0(x_1, x_2, 0) \\ \sigma_{23}^0(x_1, x_2, 0) \\ \sigma_{33}^0(x_1, x_2, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ -p \sin\beta \cos\beta \\ -p \cos^2\beta \end{bmatrix}, \qquad r < a. \tag{16}$$

The absolute value of tangential stress $\sigma_{23}^0(x_1, x_2, 0)$ is greater than it is allowed by Coulomb's law for the cracked space when $\operatorname{arctg} f < \beta < \pi/2$. This range of β is supposed in formulation of the problem. As a working hypothesis it is assumed that tangential $\sigma_{13}^0(x_1, x_2, 0)$ and normal $\sigma_{33}^0(x_1, x_2, 0)$ stresses from the solution (16) for $r < a$ are the contact stresses $\sigma_{13}^c(x_1, x_2, 0)$ and $\sigma_{33}^c(x_1, x_2, 0)$ arising on the crack. Below we shall show that this is valid.

According to Coulomb's law we find the contact tangential stress

$$\sigma_{23}^c(x_1, x_2, 0) = -fp\cos^2\beta, \qquad r < a. \tag{17}$$

Hence, the contact stress vector of perturbations, due to the crack S, is

$$\mathbf{t}(x_1, x_2, 0) = \begin{bmatrix} \sigma_{13}(x_1, x_2, 0) \\ \sigma_{23}(x_1, x_2, 0) \\ \sigma_{33}(x_1, x_2, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ p\cos\beta\,(\sin\beta - f\cos\beta) \\ 0 \end{bmatrix}, \qquad r < a. \tag{18}$$

3.2. Calculation of the relative displacements from the traction

In order to make clear the following we reproduce some fragments from [9], writing some expressions in a form which is more convenient to us and preserving the basic notations of this work: $\mathbf{u}$ is a column vector of perturbations with components u_i ($i = 1, 2, 3$), $\mathbf{G}$ is a matrix with components $G_{ij}(i, j = 1, 2, 3)$, and so on, x and ζ are (x_1, x_2) and (ζ_1, ζ_2) respectively and $\int dx$ means integration over the whole (x_1, x_2) plane if there are no other specifications.

Displacement components u_i are continuous outside S, but in S

$$[u_i]_{x_3=-0}^{x_3=+0} = b_i(x), \tag{19}$$

where $\mathbf{b}(x)$ is an unknown vector of the relative displacements of the crack surfaces. If the vector $\mathbf{t}(x)$ were known also outside S then the displacements in each half-space could be represented through the Green's matrices $\mathbf{G}^{\pm}(x, x_3)$ in the following form:

$$\mathbf{u}^+(x, x_3) = -\int \mathbf{G}^+(x - x', x_3)\mathbf{t}(x')dx', \qquad x_3 > 0,$$

$$\mathbf{u}^-(x, x_3) = \int \mathbf{G}^-(x - x', x_3)\mathbf{t}(x')dx', \qquad x_3 < 0, \tag{20}$$

because a concentrated force $\mathbf{f}$, applied to the point $x = 0$ of $(x_3 > 0)$ – half-space, produces displacements $\mathbf{G}^{+}(x, x_3)$ **f** in this half-space and analogously for $x_3 < 0$. Fourier's transform of (20) gives

$$\tilde{\mathbf{u}}^{+}(\zeta, x_3) = -2\pi\tilde{\mathbf{G}}^{+}(\zeta, x_3)\tilde{\mathbf{t}}(\zeta), \qquad x_3 > 0,$$
$$\tilde{\mathbf{u}}^{-}(\zeta, x_3) = 2\pi\tilde{\mathbf{G}}^{-}(\zeta, x_3)\tilde{\mathbf{t}}(\zeta), \qquad x_3 < 0, \tag{21}$$

where

$$\tilde{f}(\zeta) = \frac{1}{2\pi}\int f(x)\, e^{i\zeta \cdot x} dx, \tag{22}$$

$\zeta.x = \zeta_1 x_1 + \zeta_2 x_2$ and the convolution theorem is used. Substituting relations (21) into the Fourier's transform of (19) we can find

$$\tilde{\mathbf{t}}(\zeta) = -\frac{1}{2\pi}\mathbf{A}(\zeta)\tilde{\mathbf{b}}(\zeta), \tag{23}$$

where

$$\mathbf{A}(\zeta) = \left[\tilde{\mathbf{G}}^{+}(\zeta, 0) + \tilde{\mathbf{G}}^{-}(\zeta, 0)\right]^{-1}. \tag{24}$$

Applying the reverse Fourier's transform to (23), we obtain

$$\mathbf{t}(x) = -\frac{1}{4\pi^2}\int \mathbf{A}(\zeta)\tilde{\mathbf{b}}(\zeta)\, e^{-i\zeta \cdot x} d\zeta =$$
$$= -\frac{1}{8\pi^3}\int \mathbf{A}(\zeta) d\zeta \int_s \mathbf{b}(x')\, e^{i\zeta \cdot (x' - x)} dx'. \tag{25}$$

The integral over x' in (25) can be simplified by integration along $\zeta.x' = t|\zeta|$ and then by integration over t:

$$\mathbf{t}(x) = -\frac{1}{8\pi^3}\int \mathbf{A}(\zeta)\int_{-a}^{a} \check{\mathbf{b}}(t, \eta)\, e^{i|\zeta|(t - \eta.x)} dt\, d\zeta \tag{26}$$

where

$$\eta = \zeta/|\zeta|, \qquad |\zeta| = \left(\zeta_1^2 + \zeta_2^2\right)^{1/2}, \tag{27}$$

and

$$\check{\mathbf{b}}(t, \eta) = \int_s \mathbf{b}(x')\delta(t - \eta.x) dx' \tag{28}$$

is Radon's transform of the vector-function $\mathbf{b}(x')$ [10].

After changing to polar coordinates in ζ-space and integrating over $|\zeta|$, using homogeneity of the first order of matrix $\mathbf{A}(\zeta)$, the expression (26) results in

$$\mathbf{t}(x) = -\frac{1}{4\pi^2}\int_{|\eta|=1} \mathbf{A}(\eta)\mathbf{f}''^{-}(\eta.x, \eta)\mathrm{d}\eta, \tag{29}$$

where

$$\mathbf{f}(z, \eta) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\check{\mathbf{b}}(t, \eta)\mathrm{d}t}{t - z}, \tag{30}$$

$\mathbf{f}''(z, \eta)$ means $\partial^2\mathbf{f}(z, \eta)/\partial z^2$. Here the ordinary notation [5] for the limiting value of a function is used. Since the vector $\mathbf{t}(x)$ is known when $r < a$ (18), (29) is an integral equation for $\check{\mathbf{b}}(t, \eta)$.

If upper and lower half-spaces consist of the same material, then we obtain, using [9], that matrix $\mathbf{A}(\eta)$ is symmetrical and given by

$$\mathbf{A}(\eta) = \frac{\pi\mu}{1-\nu}\begin{bmatrix} 1-\nu\eta_2^2 & \nu\eta_1\eta_2 & 0 \\ \nu\eta_1\eta_2 & 1-\nu\eta_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{31}$$

We simplify equation (29), replacing the integrand by its real part, taking into consideration the symmetry of matrix $\mathbf{A}(\eta)$ and applying the Sokhotzki-Plemelj formula [5]:

$$\mathbf{t}(x) = \frac{1}{8\pi^2}\int_{|\eta|=1} \mathbf{g}''(\eta.x, \eta)\mathrm{d}\eta, \tag{32}$$

where

$$\mathbf{g}(t, \eta) = \mathbf{A}(\eta)\check{\mathbf{b}}(t, \eta), \qquad |t| < a.$$

Using the paper [9], one can find that Radon's transform of the crack-surface mutual displacement vector, which corresponds to the stress vector (18), is given by:

$$\check{\mathbf{b}}(t, \eta) = (a^2 - t^2)\frac{4(1-\nu)}{\mu(2-\nu)}\begin{bmatrix} 0 \\ -p\cos\beta\,(\sin\beta - f\cos\beta) \\ 0 \end{bmatrix}. \tag{33}$$

Now the crack-surface relative displacement vector can be found by using the reverse Radon's transform formula:

$$\mathbf{b}(x) = \frac{1}{2\pi i}\int_{|\eta|=1} \mathbf{f}'(\eta.x - 0i, \eta)\mathrm{d}\eta. \tag{34}$$

After some computations we obtain

$$\mathbf{b}(x) = \frac{8(1-\nu)(a^2 - r^2)^{1/2}}{\pi\mu(2-\nu)}\begin{bmatrix} 0 \\ -p\cos\beta\,(\sin\beta - f\cos\beta) \\ 0 \end{bmatrix}. \tag{35}$$

It should be noted that in this non-axisymmetric problem the components of the relative displacement vector b_1 and b_3 are equal to zero and b_2 depends on the polar radius but does not depend on the polar angle.

3.3. Field of displacements

Now when the vector $\breve{\mathbf{b}}(t, \eta)$ is known we find the displacement vector $\mathbf{u}(x, x_3)$. For the half-space $x_3 > 0$ we obtain from relations (21) and (23)

$$\tilde{\mathbf{u}}^+(\zeta, x_3) = \tilde{\mathbf{C}}^+(\zeta, x_3)\mathbf{A}(\zeta)\tilde{\mathbf{b}}(\zeta) = \frac{1}{2\pi}\tilde{\mathbf{G}}^+(\zeta, x_3)\mathbf{A}(\zeta)\int_S \mathbf{b}(x')\, e^{i\zeta . x'}dx', \tag{36}$$

and after integration along $\zeta . x' = t|\zeta|$ in x'-space

$$\tilde{\mathbf{u}}^+(\zeta, x_3) = \frac{1}{2\pi}\tilde{\mathbf{G}}^+(\zeta, x_3)\mathbf{A}(\zeta)\int_{-a}^{a} \breve{\mathbf{b}}(t, \eta)\, e^{i|\zeta|t}dt, \tag{37}$$

Applying the reverse Fourier's transform we obtain

$$\mathbf{u}^+(x, x_3) = \frac{1}{4\pi^2}\int \mathbf{G}^+(\zeta, x_3)\mathbf{A}(\zeta)\int_{-a}^{a} \breve{\mathbf{b}}(t, \eta)\, e^{i|\zeta|(t-\eta . x)}dt d\zeta. \tag{38}$$

Fourier's transforms of Green's matrices are available from [8]:

$$\tilde{\mathbf{G}}(\zeta, x_3) = \begin{cases} [\mathbf{B}(\zeta) + x_3\mathbf{C}(\zeta)]\, e^{-|\zeta|x_3} & (x_3 > 0) \\ [\overline{\mathbf{B}(\zeta)} - x_3\overline{\mathbf{C}(\zeta)}]\, e^{|\zeta|x_3} & (x_3 < 0) \end{cases} \tag{39}$$

where

$$\mathbf{B}(\zeta) = \frac{1}{2\pi\mu|\zeta|}\begin{bmatrix} 1-\nu+\nu\eta_2^2 & -\nu\eta_1\eta_2 & -\frac{1}{2}(1-2\nu)i\eta_1 \\ -\nu\eta_1\eta_2 & 1-\nu+\nu\eta_1^2 & -\frac{1}{2}(1-2\nu)i\eta_2 \\ \frac{1}{2}(1-2\nu)i\eta_1 & \frac{1}{2}(1-2\nu)i\eta_2 & 1-\nu \end{bmatrix},$$

$$\mathbf{C}(\zeta) = -\frac{1}{4\pi\mu}\begin{bmatrix} \eta_1^2 & \eta_1\eta_2 & -i\eta_1 \\ \eta_1\eta_2 & \eta_2^2 & -i\eta_2 \\ -i\eta_1 & -i\eta_2 & -1 \end{bmatrix}.$$

From formula (38) by changing to polar coordinates in ζ-space and integrating over $|\zeta|$ we obtain

$$\mathbf{u}^+(x, x_3) = \frac{1}{2\pi i}\int_{|\eta|=1} \mathbf{B}(\eta)\mathbf{A}(\eta)\mathbf{f}'(z, \eta)d\eta + \frac{x_3}{2\pi}\int_{|\eta|=1} \mathbf{C}(\eta)\mathbf{A}(\eta)\mathbf{f}''(z, \eta)d\eta, \tag{40}$$

where $z = \eta . x - ix_3$. It should be noted that there is an inaccuracy in analogous formula (5.4) from [9].

Substituting the necessary values in formula (40) and simplifying we obtain

$$\mathbf{u}^+(x, x_3) = \alpha\int_0^{2\pi}\left\{\begin{bmatrix}0\\1-\nu\\\frac{1}{2}i(1-2\nu)\eta_2\end{bmatrix}\left[(\eta.x - ix_3)\ln\frac{\eta.x - ix_3 - a}{\eta.x - ix_3 + a} + 2a\right]\right.$$

$$\left. + \frac{x_3}{2i}\begin{bmatrix}\eta_1\eta_2\\\eta_2^2\\-i\eta_2\end{bmatrix}\left[\ln\frac{\eta.x - ix_3 - a}{\eta.x - ix_3 + a} + \frac{2a(\eta.x - ix_3)}{(\eta.x - ix_3)^2 - a^3}\right]\right\}\mathrm{d}\phi, \qquad (41)$$

where

$$\alpha = \frac{-p\cos\beta\,(\sin\beta - f\cos\beta)}{\pi^2\mu(2-\nu)}, \qquad \eta = \begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}\cos\phi\\\sin\phi\end{bmatrix}.$$

Similarly, for half-space $x_3 < 0$, we find

$$\mathbf{u}^-(x, x_3) = \alpha\int_0^{2\pi}\left\{-\begin{bmatrix}0\\1-\nu\\-\frac{1}{2}i(1-2\nu)\eta_2\end{bmatrix}\left[(\eta.x + ix_3)\ln\frac{\eta.x + ix_3 - a}{\eta.x + ix_3 + a} + 2a\right]\right.$$

$$\left. + \frac{x_3}{2i}\begin{bmatrix}\eta_1\eta_2\\\eta_2^2\\i\eta_2\end{bmatrix}\left[\ln\frac{\eta.x + ix_3 - a}{\eta.x + ix_3 + a} + \frac{2a(\eta.x + ix_3)}{(\eta.x + ix_3)^2 - a^2}\right]\right\}\mathrm{d}\phi. \qquad (42)$$

The integrals in (41) and (42) can be calculated after reducing them to a form, suitable for application of the theorem of residues. Let us calculate for example, $u_2^+(x_1, x_2, x_3)$. Changing to a complex variable $\omega = \mathrm{e}^{i\phi}$ we obtain

$$u_2^+(x_1, x_2, x_3)$$

$$= \alpha\int_{|\omega|=1}\left\{\ln\frac{(\omega-\omega_1)(\omega-\omega_2)}{(\omega-\omega_3)(\omega-\omega_4)}\left[-(1-\nu)\left(\frac{ix_1+x_2}{2} + \frac{x_3}{\omega} + \frac{ix_1-x_2}{2\omega^2}\right)\right.\right.$$

$$\left. + \frac{x_3}{8}\left(\omega - \frac{2}{\omega} + \frac{1}{\omega^3}\right)\right] + \frac{2a(1-\nu)}{i\omega}$$

$$\left. - \frac{ax_3}{2i(ix_1+x_2)^2}\left(\omega^2 - 2 + \frac{1}{\omega^2}\right)\frac{\omega^2(ix_1+x_2) + 2x_3\omega + ix_1 - x_2}{(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)}\right\}\mathrm{d}\omega, \quad (43)$$

$$\omega_{1,3} = \frac{A - x_3 \pm i(a-B)}{ix_1 + x_2}, \qquad \omega_{2,4} = \frac{-A - x_3 \pm i(B+a)}{ix_1 + x_2},$$

$$A, B = \left(\left(\left(\left(x_1^2 + x_2^2 + x_3^2 - a^2 \right)^2 + 4a^2 x_3^2 \right)^{1/2} \pm \left(x_1^2 + x_2^2 + x_3^2 - a^2 \right) \right) / 2 \right)^{1/2}.$$

We investigate a disposition of the singular points relatively to the integration contour. With exception of the following cases:

(1) $x_3 = 0$

and

(2) $x_1 = x_2 = 0$ simultaneously,

we obtain that

$$|\omega_1| < 1, \qquad |\omega_3| < 1, \qquad |\omega_2| > 1, \qquad |\omega_4| > 1. \tag{44}$$

The integral

$$J(\omega_1, \omega_2, \omega_3, \omega_4) = x_3(\nu - 1.25) \int_{|\omega|=1} \frac{1}{\omega} \ln \frac{(\omega - \omega_1)(\omega - \omega_2)}{(\omega - \omega_3)(\omega - \omega_4)} \mathrm{d}\omega \tag{45}$$

from (43) can be calculated by means of investigation of its partial derivatives. Using the values of ω_2 and ω_4 from (43), we obtain

$$J = x_3(4\nu - 5)\pi \left(\operatorname{arctg} \frac{B + a}{A + x_3} - \pi m \right),$$

where the unknown natural number m is still to be determined. We get rid of other logarithmic addenda by integrating them by parts. Then we find all other integrals by means of the theorem of residues, remembering that from the singular points only ω_1, ω_3 and $\omega = 0$ are located in domain $|\omega| = 1$.

After performing the following substitutions

$$\omega_1 = \frac{w_1}{ix_1 + x_2}, \qquad \omega_3 = \frac{\bar{w}_1}{ix_1 + x_2}, \qquad \omega_2 = \frac{w_2}{ix_1 + x_2}, \qquad \omega_4 = \frac{\bar{w}_2}{ix_1 + x_2},$$

where a bar over w means complex conjugation, the real part of the expression for $u_2^-(x_1, x_2, x_3)$ can be separated. From the condition of vanishing of the displacement at infinity we find the constant $m = 0$. Finally

$$u_2^+(x_1, x_2, x_3) = \alpha\pi \left\{ x_3(4\nu - 5) \operatorname{arctg} \frac{B + a}{A + x_3} + 2(1 - \nu)(B - a) \right.$$

$$- 2(1 - \nu) \frac{(B - a)\left(x_1^2 + x_2^2\right)}{(A - x_3)^2 + (B - a)^2} - \frac{x_3\left(x_2^2 - x_1^2\right)(A - x_3)(B - a)}{2\left(x_1^2 + x_2^2\right)^2}$$

$$-\frac{x_3(x_2^2-x_1^2)(A-x_3)(B-a)}{2\left[(A-x_3)^2+(B-a)^2\right]^2}-\frac{ax_3(x_2^2-x_1^2)}{2(x_1^2+x_2^2)(A^2+B^2)}$$

$$\times\left[2B(A-x_3)(B-a)+A\left((A-x_3)^2-(B-a)^2\right)\right]+\frac{ax_3A}{A^2+B^2}$$

$$+\frac{ax_3(x_2^2-x_1^2)}{2\left[(A-x_3)^2+(B-a)^2\right]^2(A^2+B^2)}$$

$$\left.\times\left[2B(A-x_3)(B-a)-A\left((A-x_3)^2-(B-a)^2\right)\right]\right\}. \tag{46}$$

Similarly, from the expression (41) we obtain

$$u_3^+(x_1,x_2,x_3)=\alpha\pi x_2\left\{\frac{(B-a)\left[A(1-2\nu)-x_3(1+2\nu)\right]}{2(x_1^2+x_2^2)}\right.$$

$$+\frac{2\nu x_3(B-a)}{(A-x_3)^2+(B-a)^2}+\frac{(1-2\nu)(x_1^2+x_2^2)(B-a)(A-x_3)}{2\left[(A-x_3)^2+(B-a)^2\right]^2}$$

$$+(1-2\nu)\operatorname{arctg}\frac{B+a}{A+x_3}-\frac{ax_3\left[A(A-x_3)+B(B-a)\right]}{(A^2+B^2)(x_1^2+x_2^2)}$$

$$\left.+\frac{ax_3\left[A(A-x_3)-B(B-a)\right]}{(A^2+B^2)\left[(A-x_3)^2+(B-a)^2\right]}\right\} \tag{47}$$

and

$$u_1^+(x_1,x_2,x_3)=\frac{\alpha\pi a x_1x_2x_3}{B-a}\left\{\frac{1}{(x_1^2+x_2^2)^2}\operatorname{Im}\left[\frac{w_1^4}{(w_1-w_2)(w_1-\bar{w}_2)}\right]\right.$$

$$+\frac{4x_3}{(x_1^2+x_2^2)^2}\operatorname{Im}\left[\frac{w_1^3}{(w_1-w_2)(w_1-\bar{w}_2)}\right]$$

$$-\frac{3}{x_1^2+x_2^2}\operatorname{Im}\left[\frac{w_1^2}{(w_1-w_2)(w_1-\bar{w}_2)}\right]$$

$$+3\,\mathrm{Im}\left[\frac{1}{(w_1-w_2)(w_1-\bar{w}_2)}\right]+4x_3\,\mathrm{Im}\left[\frac{1}{w_1(w_1-w_2)(w_1-\bar{w}_2)}\right]$$

$$-\left(x_1^2+x_2^2\right)\mathrm{Im}\left[\frac{1}{w_1^2(w_1-w_2)(w_1-\bar{w}_2)}\right]\Bigg\}\Bigg). \tag{48}$$

In accordance with formulas (41) and (42) the displacements $u_1^-(x_1, x_2, x_3)$ and $u_2^+(x_1, x_2, x_3)$ can be obtained from $u_1^+(x_1, x_2, x_3)$ and $u_2^+(x_1, x_2, x_3)$ respectively by multiplying them by -1 and by substituting $-x_3$ instead of x_3; the displacement $u_3^-(x_1, x_2, x_3)$ can be obtained from $u_3^+(x_1, x_2, x_3)$ by substituting $-x_3$ instead of x_3. In particular, from (46), (47), (48) and the similar formulas we obtain

$$u_1^\pm(x_1, x_2, 0)=u_2^\pm(x_1, x_2, 0)=0, \qquad (r\geqslant a)$$

$$u_3^\pm(x_1, x_2, 0)=\alpha\pi(1-2\nu)x_2\left[\mathrm{arctg}\left(a(r^2-a^2)^{-1/2}\right)-a(r^2-a^2)^{1/2}r^{-2}\right]; \tag{49}$$

$$u_1^\pm(x_1, x_2, 0)=0,$$

$$u_2^\pm(x_1, x_2, 0)=\pm4\alpha\pi(1-\nu)(a^2-r^2)^{1/2}, \qquad (r\leqslant a)$$

$$u_3^\pm(x_1, x_2, 0)=\alpha\pi(1-2\nu)x_2\frac{\pi}{2}. \tag{50}$$

The following expressions

$$u_1^\pm(0, 0, x_3)=u_3^\pm(0, 0, x_3)=0,$$

$$u_2^\pm(0, 0, x_3)=\pm\alpha\pi\left[(5-4\nu)x_3\left(\mathrm{arctg}\frac{x_3}{a}\mp\frac{\pi}{2}\right)+4a(1-\nu)+\frac{ax_3^2}{a^2+x_3^2}\right]$$

can be obtained more simply directly from formulas (41) and (42).

According to formulas (50) the mutual displacement in x_1-direction and the mutual penetration in x_3-direction of the crack surfaces are absent. Hence, the working hypothesis, adopted in 3.1, is valid. The sum of displacements $\mathbf{u}^0(x_1, x_2, x_3)$ and $\mathbf{u}(x_1, x_2, x_3)$ is the solution to the problem on compression of the cracked elastic space that we wanted to find. Now the idea of Section 2 can be extended to the 3-D case.

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